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Implementation of Strength-Based Failure Criteria in the Lamination Parameter Design Space

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Lamination parameters have proven to be exceedingly useful for the efficient design optimization of composite structures due to their continuous nature and flexibility in the final choice of stacking sequence with a reduced number of design variables. So far, failure criteria could only be incorporated into the lamination parameter design space for a predefined combination of fiber angles. This drastically limits the design space available to the designer. This paper investigates incorporating the Tsai–Wu failure criterion into the lamination parameter design space in the most general setting. A conservative failure envelope is derived that guarantees a failure-free region of the lamination parameter space regardless of the fiber orientations in the laminate stacking sequence. The proposed approach allows strength criteria to be included both as a constraint as well as an objective function during the optimization process. The derived strength envelope is then used to compare strength- and stiffness-based designs. The results clearly indicate that the degree of correlation between stiffness- and strength-driven designs depends on the properties of the material under consideration.

I. Introduction

LAMINATION parameters, first introduced by Tsai and Pagano [1] and Tsai and Hahn [2], provide a compact notation for the description of the stiffness properties of a laminate lay-up configuration. The complete set of stiffness matrices of any laminate can be described as a linear function in terms of only 12 lamination parameters and, for the important special case of a symmetric balanced laminate, only four lamination parameters are necessary. When lamination parameters are used as design variables for stacking sequence design, the number of design variables is independent of the number of layers, thus reducing the complexity of the optimization problem. Moreover, the feasible region of the lamination parameter design space is convex. Several design problems have also been shown to be convex in the lamination parameter design space [3–5], leading to further simplification of numerical optimization methods. Once optimal lamination parameters are determined, the stacking sequence can be constructed (e.g., using a genetic algorithm [6,7]), which provides the designer with a range of suitable laminate designs, all sharing the same performance, to choose from.

The application of lamination parameters for structural optimization has been successful for a variety of optimization problems. To name a few, Miki and Sugiyama [8] presented a graphical approach for maximizing structural stiffness, as well as buckling and fundamental frequency. Fukunaga and Vanderplaats [9] investigated stiffness optimization using lamination parameters. Fukunaga and Sekine later presented works using lamination parameters to optimize stiffness [10] and fundamental frequency [3], as well as buckling [4]. Hammer et al. [11] and Setoodeh et al. [12] formulated stiffness optimization in terms of minimum compliance where the ply lay up varied spatially. Foldager et al. [5] used a combination of ply angles, ply thickness, and lamination parameters to force a convex stiffness design optimization formulation.

Lamination parameters are generally applicable to any laminates for which classical lamination theory is valid. In general, lamination parameters are not applicable to general 3-D composite materials (currently, an extension of lamination parameters such as quantities to 3-D composites is under consideration by the authors). There have been three main difficulties in the use of lamination parameters. The first problem has been that the definition of the design space in the case of a generic laminate has been a difficult issue to tackle. However, design problems requiring both in-plane and out-of-plane lamination parameters can be solved using approximate feasible regions, as presented by Setoodeh et al. [13]. The second problem has been the perceived difficulty of recovering practical lay ups from lamination-parameter-based design optimization. This was treated satisfactorily in a number of publications by Autio [7] and Diaconu et al. [14], as well as in work by the authors [15] in which a genetic algorithm is used to match a suitable stacking sequence to the optimal lamination parameters.

The third problem with using lamination parameters has been the difficulty with which strength constraints could be incorporated into the design process. Strength constraints based on popular failure criteria such as the Tsai–Wu criterion [16] depend not only on the stiffness properties, but also explicitly on ply angles, precluding the use of lamination parameters. Thus, strength constraints could be incorporated in lamination-parameter-based optimization formulation only for the special case of ply angles restricted to a predetermined discrete set, as presented by Gürdal et al. [17]. In a paper on using lamination parameters for reliability-based optimization, Kogiso et al. [18] also used the Tsai–Wu failure criterion for a fixed set of ply angles, relating ply strength via a strength ratio to the maximum allowable strain for that ply. Additionally, Groenwold and Haftka [19] investigated the optimization for strength, limiting the laminate to a single orientation angle.

This paper investigates the incorporation of the Tsai–Wu failure criterion into the lamination parameter design space in the most general setting. The main idea is to map the Tsai–Wu failure criterion to strain space; the usefulness of this approach was first demonstrated by Nakayasu and Maekawa [20]. When written in terms of strain components in global coordinates (rather than material coordinates), the ply angle explicitly appears in the failure criterion. A conservative failure envelope is constructed by finding the region in

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the strain space that is safe regardless of the ply angle. An analytic solution is obtained for this conservative failure envelope. It is shown that two different envelope equations may apply depending on material stiffness properties and failure stresses. It is also shown that the envelope equation is a function of only two strain invariants. However, the proposed failure envelope should not be considered as a strain invariant failure criterion such as the one proposed by Gosse and Christensen [21].

Stiffness-based optimization is relatively straightforward to implement and, hence, is often used as a substitute for strength-based design. Therefore, it is interesting to compare the strength- and stiffness-based optimal designs. The objective of the strength-based optimization is to minimize the failure index, which is equivalent to maximizing the factor of safety, as proposed by Groenwold and Haftka [19]. The objective of the stiffness-based optimization is to minimize compliance. The optimization is carried out for several different materials, for a range of stiffness ratios (E_1/E_2), and for a combination of axial and shear loading. Results show that the correlation between stiffness- and strength-driven designs is generally favorable but depends on the properties of the material under consideration and the type of loading.

In the next section a brief introduction to lamination parameters is given, followed in Sec. III by the formulation of a failure envelope independent of ply orientation. The envelope is subsequently used to derive an expression for the failure index in Sec. IV. In Sec. V a method to optimize structural stiffness is briefly discussed. Finally, the results for several materials are compared with those found for maximum strength in Sec. VI, followed by closing remarks.

II. Lamination Parameters

The in-plane lamination parameters as initially introduced by Tsai and Pagano [1] and Tsai and Hahn [2] are defined as

$$(V_1, V_2, V_3, V_4) = \int_{-1/2}^{1/2} (\cos 2\theta(\bar{z}), \sin 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 4\theta(\bar{z}))\,\mathrm{d}\bar{z} \tag{1}$$

where $\bar{z} = z/h$ is the normalized through-the-thickness coordinate of the layers [22], h is the total thickness of the laminate, and $\theta(\bar{z})$ is the fiber angle at $\bar{z}$. The in-plane laminate stiffness matrix $\mathbf{A}$ then becomes a linear function of the lamination parameters as follows:

$$\mathbf{A} = h(\Gamma_0 + V_1\Gamma_1 + V_2\Gamma_2 + V_3\Gamma_3 + V_4\Gamma_4) \tag{2}$$

where the Γ_i $(i = 1, \ldots, 4)$ matrices in terms of the material invariants [22] are given by

$$\Gamma_0 = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} U_2 & 0 & 0 \\ 0 & -U_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & U_2/2 \\ 0 & 0 & U_2/2 \\ U_2/2 & U_2/2 & 0 \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} U_3 & -U_3 & 0 \\ -U_3 & U_3 & 0 \\ 0 & 0 & -U_3 \end{bmatrix}, \qquad \Gamma_4 = \begin{bmatrix} 0 & 0 & U_3 \\ 0 & 0 & -U_3 \\ U_3 & -U_3 & 0 \end{bmatrix}$$

The material invariants, U_i for $i = 1, \ldots, 5$, are defined in Appendix A. The lamination parameters are not independent because the trigonometric functions used in Eq. (1) are related. Considering the trigonometric dependency, the feasible domain for the in-plane lamination parameters is known to be defined by [11]

$$\begin{gathered} 2V_1^2(1 - V_3) + 2V_2^2(1 + V_3) + V_3^2 + V_4^2 - 4V_1V_2V_4 \le 1 \\ V_1^2 + V_2^2 \le 1 \qquad -1 \le V_i \le 1 \qquad (i = 1, \ldots, 4) \end{gathered} \tag{3}$$

According to the definition of the lamination parameters in Eq. (1), only two of the lamination parameters, namely, V_1 and V_3, are required to fully model the in-plane stiffness of balanced symmetric laminates in which $V_2 = V_4 = 0$. This would simplify the aforementioned set of inequality constraints to

$$V_3 \ge 2V_1^2 - 1 \qquad -1 \le V_i \le 1 \qquad (i = 1, 3) \tag{4}$$

In Sec. III, the Tsai–Wu failure criterion is used to define a design envelope in which laminate failure is no longer dependent on the individual ply orientations.

III. Failure Envelope

The Tsai–Wu failure criterion is a well-known first-ply failure criterion defined as [16]

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 = 1 \tag{5}$$

where F_i and F_{ij} are the second- and fourth-order strength tensors, with $i, j = 1, 2, \ldots, 6$, and are given in Appendix A. The material stresses are related to the material strains via the reduced stiffness matrix $\mathbf{Q}$ [16]. Allowing the failure criteria to be written in terms of the components of the material strain tensor

$$G_{11}\epsilon_1^2 + G_{22}\epsilon_2^2 + G_{66}\epsilon_{12}^2 + G_1\epsilon_1 + G_2\epsilon_2 + 2G_{12}\epsilon_1\epsilon_2 = 1 \tag{6}$$

where the strain coefficients, G_{ij}, are defined in Appendix A. The material strains $(\epsilon_1, \epsilon_2, \epsilon_{12})$ can subsequently be related to the laminate strains $(\epsilon_x, \epsilon_y, \epsilon_{xy})$ using the following transformation matrix [16]:

$$\begin{bmatrix} \frac{1}{2}(1+c) & \frac{1}{2}(1-c) & s \\ \frac{1}{2}(1-c) & \frac{1}{2}(1+c) & -s \\ -\frac{1}{2}s & \frac{1}{2}s & c \end{bmatrix} \tag{7}$$

where $s = \sin(2\theta)$ and $c = \cos(2\theta)$. Substituting the transformed strains of Eq. (7) into the failure envelope Eq. (6), we obtain the failure envelope equation in terms of laminate strains and the ply angle in the form

$$F(\epsilon_x, \epsilon_y, \epsilon_{xy}, s, c) = 0 \tag{8}$$

The objective is to construct a design envelope within which no failure occurs regardless of the ply orientation. To this end, we construct the geometric "envelope," which is defined as the surface tangent to the family of failure surfaces, Eq. (8), parameterized using the ply angle θ. The equation for the envelope is given by

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = 0 \tag{9}$$

which can be expanded using the chain rule as

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = c\frac{\partial F}{\partial s} - s\frac{\partial F}{\partial c} = 0 \tag{10}$$

Both F and F_θ are polynomial functions of c and s. Because s and c are dependent upon the ply angle θ, they are not independent but have to satisfy the trigonometric relation

$$s^2 + c^2 - 1 = 0 \tag{11}$$

The equation for the failure envelope is obtained by eliminating s and c between Eqs. (8), (10), and (11). The elimination is achieved by using Dixon's resultant [23] for the elimination of polynomial equations. For the sake of brevity, the algebraic details of the application of Dixon's resultant are omitted; however, commercial mathematics software, such as Mathematica™ [24], provides packages that implement Dixon's resultant. The result yields the following two Eqs. (12) and (13), each representing a surface traced out by the failure criteria for all ply orientations:

$$\begin{aligned} &4u_6^2I_2^2 - 4u_6u_1I_2^2 + 4\left(1 - u_2I_1 - u_3I_1^2\right)(u_1 - u_6) \\ &\quad + (u_4 + u_5I_1)^2 = 0 \end{aligned} \tag{12}$$

$$u_1^2 I_2^4 - I_2^2(u_4 + u_5 I_1)^2 - 2u_1 I_2^2\left(1 - u_2 I_1 - u_3 I_1^2\right) + \left(1 - u_2 I_1 - u_3 I_1^2\right)^2 = 0 \tag{13}$$

where I_1 is the volumetric strain invariant and I_2 is the maximum shear strain [25] given by

$$I_1 = \epsilon_x + \epsilon_y, \qquad I_2 = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2} \tag{14}$$

The terms u_i, $i = (1, \ldots, 6)$ are defined in terms of the strain coefficients of Eq. (6) as listed in Appendix A. It should be clear that the feasible design space described by Eqs. (12) and (13) is material dependent, because u_i is a function of strain coefficients, G_{ij}, which is a function of the reduced stiffness matrix **Q** and material strength coefficients F_i, F_{ij}. It should also be noted that the failure envelopes prescribed by Eqs. (12) and (13) represent a conservative approximation of the Tsai–Wu failure criterion in terms of strain invariants and should not be confused with a strain invariant failure criterion such as that presented by Gosse and Christensen [21] and, hence, should not be considered as a new failure criterion.

Upon inspection, the first envelope (12) is a second-order equation with respect to the strains, and the second envelope (13) is of the fourth order. These two envelope equations do not intersect one another, but may become tangent, as shown in Fig. 1. The safe region is the region common to the Tsai–Wu failure envelope for all angles. As such, the envelope equation describing the inner envelope is always to be used. Whether the inner envelope is represented by the second- or fourth-order equations is dependent on the properties of the material under consideration. When the fourth-order equation describes the inner envelope, it is usually factorable into the product of two equations leading to a self-intersecting nonsmooth envelope.

To gain a better understanding of the feasible design envelopes, they are plotted for specific materials. As an example, consider the materials listed in Appendix B, Table B1, which have stiffness ratios, E_1/E_2, ranging from approximately 9 to 17. In Fig. 1, the actual strain envelope for the various material orientation angles are plotted together with the two curves prescribed by the derived equations. In this case, ϵ_{xy} has been set to zero; however, similar results can be generated for a range of ϵ_{xy} values. It is clear from the figure that in each case one of the two equations accurately prescribes the inner strain envelope, which is in fact independent of the fiber orientation. A method of selecting the critical envelope equation is treated in Sec. IV.

As can be seen in Fig. 1, the conservative design envelopes prescribed by Eqs. (12) and (13) are convex in the strain space, as they are the intersection of the infinite number of convex sets defined by the Tsai–Wu failure criterion. In Sec. IV, a formulation for the derived design envelope in terms of a safety factor is presented.

IV. Formulation of a Strength Constraint

Equations (12) and (13) represent the failure envelope in the strain design space. To simplify the applicability as a constraint, or alternatively as an objective function, for optimization, the equations of the design envelope can be reformulated in terms of the safety factor, λ, which can be defined as

$$\lambda = \frac{b}{a} \tag{15}$$

where a is the distance between the origin and an arbitrary point P in the feasible design space, and b is the length of the vector from the origin through point P to the point on the envelope boundary, P^*, as illustrated in Fig. 2.

In essence, λ is a scaling factor that, when applied to the values of $\boldsymbol{\epsilon}$ at a generic point P representing applied strains, gives the values of $\boldsymbol{\epsilon}^*$ at the corresponding point on the boundary, P^*. Therefore, the strain invariants, I_1 and I_2, can be related to those at the boundary of the failure envelopes by substituting $I_1^* = \lambda I_1$ and $I_2^* = \lambda I_2$ into the failure envelope Eqs. (12) and (13), yielding two polynomials in terms of λ:

$$\begin{aligned} f_1(\lambda) &= a_{12}\lambda^2 + a_{11}\lambda + a_{10} \\ f_2(\lambda) &= a_{24}\lambda^4 + a_{23}\lambda^3 + a_{22}\lambda^2 + a_{21}\lambda + a_{20} \end{aligned} \tag{16}$$

where the coefficients a_{ij} are functions of the strain invariants I_1 and I_2 and are listed in Appendix C. Solving for λ yields up to six roots; the equation with the smallest positive real root represents the active envelope, because the smallest safety factor is critical. The active envelope is not independent of the strains. It is also noted that the fourth-order envelope self-intersects and can therefore be thought of as two smooth curves, as can be seen in Fig. 1c. Corresponding to each of these two smooth curves is a positive root λ, which is a continuous function of strains.

Groenwold and Haftka [19] first suggested using a factor of safety for strength optimization by directly maximizing λ. However, λ is not differentiable near the origin, and this can lead to numerical problems. To remedy this, the failure index, $r(\boldsymbol{\epsilon})$, is defined as the inverse of the factor of safety squared, which guarantees differentiability at all points within the failure envelope:

$$r(\boldsymbol{\epsilon}) = \frac{1}{\lambda^2} \tag{17}$$

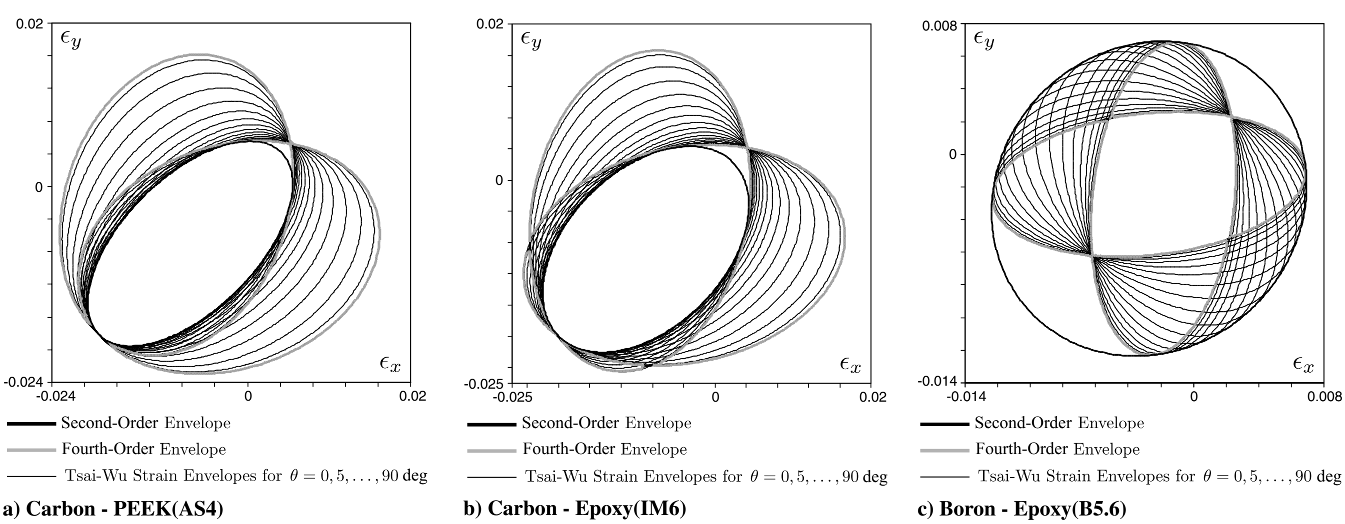


Fig. 1 Strain envelopes for various fiber orientation angles, including the second- and fourth-order solutions derived in Eqs. (12) and (13) (ϵ_x vs ϵ_y, with $\epsilon_{xy} = 0$).

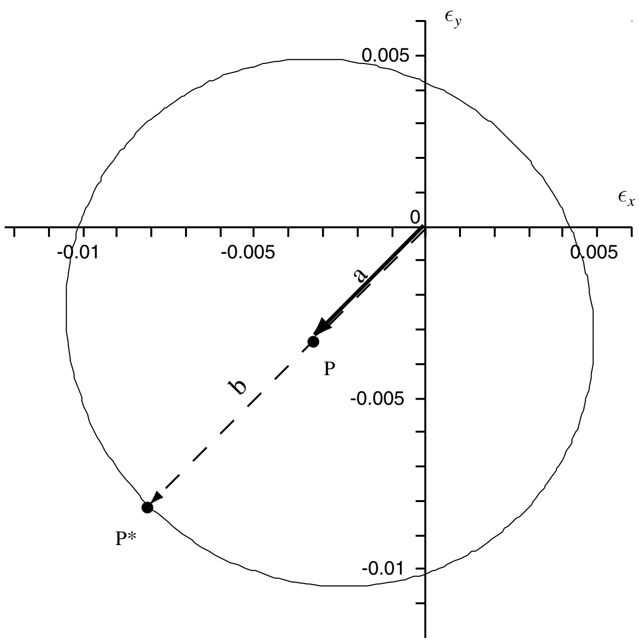


Fig. 2 Definition of λ with respect to an arbitrary point, *P*, within the design envelope and the corresponding point, *P, on the boundary.**

where λ is the smallest positive real root obtained from Eq. (16). If the implemented failure criterion is quadratic (such as Tsai–Hill), the failure index, r, is identical to the failure criterion. Otherwise, the failure index is representative of the safety factor and is a function that has a value of 0 at the origin, when no strain is present, and a value of 1 at the envelope boundary, which indicates laminate failure. Hence, the strength constraint can be formulated as

$$r(\boldsymbol{\epsilon}) - 1 \leq 0 \tag{18}$$

which is valid at any point within the laminate and is only a function of the local strain. The strain at any point, (x, y, z), can be related to the midplane strains, $\boldsymbol{\epsilon}^0$, and the curvatures $\boldsymbol{\kappa}$, as follows:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0(x, y) + z\boldsymbol{\kappa}(x, y) \tag{19}$$

where $z \in [-\frac{h}{2}, \frac{h}{2}]$ is the thickness coordinate, and x and y are panel coordinates. For convenience of use in the optimization of thin plates and shells, it is advantageous to eliminate the dependency on thickness coordinate z in the formulation of the strength constraint. This can be achieved by stipulating that the worst-case value is still safe. Thus, the strength constraint at any point (x, y) in the laminate becomes

$$\max_z(r) - 1 \leq 0 \tag{20}$$

as the largest values of r on interval $z \in [-\frac{h}{2}, \frac{h}{2}]$ will be critical. The function r is a convex function of the strains, $\boldsymbol{\epsilon}$, which itself is a linear function in z. Because all ply orientations are present at any through-the-thickness point, it follows from the properties of convex functions that r is a unimodal function of z with a unique minimum and, hence, the maximum will have to occur at one of the extreme fibers, that is, $z = -\frac{h}{2}$ or $z = \frac{h}{2}$. This result may seem paradoxical at first because it is well known that the most critical point through the thickness of a composite laminate need not be one of the extreme fibers. This counterintuitive finding can be explained by noting that in the derivation of the failure envelope it is assumed that any ply orientations are possible at any given point. The conservative strain constraint, Eq. (18), does not take into account the actual order of the layers in the laminate. It can be anticipated that this approach can be excessively conservative for bending dominated problems, a fact that will be confirmed by the numerical results in Sec. VI.

For pure in-plane strains (e.g., in-plane loading of a symmetric laminate), the strains are constant through the thickness, and the strength constraint, Eq. (18), can be directly applied. On the other hand, when bending curvatures are present the strength constraint can be replaced by two constraints:

$$r^+ - 1 \leq 0 \quad \text{and} \quad r^- - 1 \leq 0 \tag{21}$$

where $r^{\pm} = r(x, y, \pm h/2)$.

V. Optimization Formulation

In this section, the optimization problems for designing a panel under constant in-plane loads for maximum stiffness and maximum strength are formulated. The availability of both formulations makes it straightforward to compare the results of strength- and stiffness-optimized panels and provides a useful test for the intuitive notion that stiffness, which is far easier in mathematical treatment, is a good surrogate objective function for strength.

In Sec. V.A, a method of approximating the failure index developed in Sec. IV is presented to efficiently optimize for laminate strength. Subsequently, a corresponding formulation for stiffness optimization is presented.

A. Strength Optimization

To maximize structural strength, Groenwold and Haftka [19] proposed the maximization of the factor of safety λ, which is equivalent to the minimization of the failure index r. In Sec. IV, an expression for the failure index was developed as a function of the laminate strains, $\boldsymbol{\epsilon}$. For the in-plane problem, the design variables for the optimization are the lamination parameters. The strains can be expressed as functions of the design variables using classical lamination theory,

$$\boldsymbol{\epsilon} = \mathbf{S}(\mathbf{V}) \cdot \mathbf{N} \tag{22}$$

where $\mathbf{S}$ is the compliance matrix, which is the inverse of the in-plane stiffness matrix $\mathbf{A}$, defined in terms of lamination parameters, V_i, in Eq. (2). Note that, to maintain consistency with the stiffness optimization formulation derived in Sec. V.B, $\boldsymbol{\epsilon}$ refers to the engineering strain vector. Thus, the strength-optimization problem can be formulated as

$$\min_{V_i} r(\boldsymbol{\epsilon}(V_i)) \tag{23}$$

subject to the constraints imposed by the lamination parameter design space [Eqs. (3) or (4)].

To efficiently solve Eq. (23), an approach is proposed in which the failure index, r, is approximated as a function of the lamination parameters. By approximating r, excessive evaluations of the failure index itself are avoided. In most practical design optimization problems, evaluation of the failure index requires a computationally expensive finite element analysis. For this reason, it is customary to use an approximation in structural optimization [26] to keep the number of function evaluations to a minimum.

It can be shown that the failure index, r, is a homogeneous function of the second order with respect to the strains [19]. Therefore, it can be approximated using a second-order Taylor series expansion in terms of strains:

$$r(\boldsymbol{\epsilon}) \approx r^{(k)} + \mathbf{g}^{(k)^T} \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(k)}) + \frac{1}{2!}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(k)})^T \cdot \mathbf{H}^{(k)} \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(k)}) \tag{24}$$

with

$$\mathbf{g}^{(k)} = \left.\frac{\partial r}{\partial \boldsymbol{\epsilon}}\right|_{\epsilon=\epsilon^k} \quad \text{and} \quad \mathbf{H}^{(k)} = \left.\frac{\partial^2 r}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\epsilon}}\right|_{\epsilon=\epsilon^k} \tag{25}$$

Using Euler's theorem [27] of homogeneous functions, it can be shown that $\mathbf{g}^{(T)} \cdot \boldsymbol{\epsilon} = 2r$ and $\mathbf{H} \cdot \boldsymbol{\epsilon} = \mathbf{g}$; hence, the approximation simplifies to

$$r(\boldsymbol{\epsilon}) \approx \tfrac{1}{2}\boldsymbol{\epsilon}^T \cdot \mathbf{H}^{(k)} \cdot \boldsymbol{\epsilon} = \tfrac{1}{2}(\mathbf{S} \cdot \mathbf{N})^T \cdot \mathbf{H}^{(k)} \cdot (\mathbf{S} \cdot \mathbf{N}) \tag{26}$$

where **H** can be derived analytically using the chain rule as shown in Appendix C. The lamination parameters corresponding to the minimum value for the approximation of r can be found using any standard optimization algorithm.

B. Stiffness Optimization

Maximum stiffness designs can also be computed using lamination parameters by minimizing the compliance [12]. The minimization problem can be formulated as

$$\min_{\mathbf{V}} \tfrac{1}{2}\mathbf{N}^T \cdot \mathbf{S}(\mathbf{V}) \cdot \mathbf{N} \tag{27}$$

subject to the constraints imposed by the lamination parameter design space as described in Sec. II. The compliance is the measure of the complementary work done by the external loads and has been shown to be convex [12].

VI. Results

The two optimization formulations developed in Secs. V.A and V.B will be applied to several design problems. The purpose of the numerical tests is to 1) assess the range of behavior of the strength-optimization theory for different materials, and 2) discuss the relationship between strength- and stiffness-optimized designs. Three different materials are considered, as listed in Appendix B. All examples consider the following load case of combined axial and shear loading:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} 1-\omega \\ 0 \\ \omega \end{Bmatrix} \cdot N \tag{28}$$

with $\omega \in [0, 1]$, where $\omega = 0$ represents pure axial tension or compression depending on the sign of N, and $\omega = 1$ represents pure shear loading. N was given a numerical value of $\pm 150 \times 10^6$ N/m. This is an arbitrary choice but provides a good range for r for the various materials.

In the rest of this section, numerical results will be used to assess the behavior of the developed failure index. In Sec. VI.A, the conservativeness of the strength formulation is quantified. The proposed optimization formulation is verified for the three materials under consideration in Sec. VI.B. In Sec. VI.C, the optimization procedure is executed for both balanced and unbalanced laminates for a range of tensile, compressive, and shear loads.

A. Strength Envelope Conservativeness

The proposed failure index, r, assumes that all ply orientations are present in the laminate. In a real laminate lay up this may not be the case. Hence, if the ply orientations that define the critical boundary of the failure envelope are not present, the formulation could lead to conservative designs. Two particular cases are likely to exhibit this conservative behavior. The first is when all fibers are aligned in a single direction, such as under uniaxial loading. And the second is when there is a strong variation of the strain through the laminate thickness, such as when bending is dominant. The first case corresponds to pure tension, $\omega = 0$ in Eq. (28), $N_x = N$, and the second case to axial bending, M_x, which are most likely to be the worst-case scenarios.

To assess the conservativeness, the safety factor λ, defined using the envelope, is compared with the safety factor predicted by the Tsai–Wu failure criterion, λ_{TW}. The ratio of the safety factors, $\lambda_{\mathrm{TW}}/\lambda$, for several laminates are presented in Table 1. For all three materials investigated, the safety factor is significantly underestimated when considering the intuitively optimal 0 deg design for pure tension. When considering other laminates with different ply orientations present, the difference between the two safety is factors reduced. In fact, for the axially loaded laminate, $[\pm 45, 0_4, 90_2]_s$, the safety factors only differ between 0 and 12%.

If the same laminate, $[\pm 45, 0_4, 90_2]_s$, is considered under bending loads, the difference between the lamination-based failure envelope predictions and the Tsai–Wu failure criterion can be as large as 33%. This is due to the fact that the relative through-the-thickness position of plies with different orientations influences both the safety factor for the derived envelope as well as the Tsai–Wu failure criterion. For example, the plies can be rearranged to obtain a laminate with the same in-plane lamination parameters but with different surface plies, $[0_4, \pm 45, 90_2]_s$, yielding different values for the conservativeness of the design. Hence, the degree of conservativeness is clearly influenced by the chosen stacking sequence. Because the envelope cannot explicitly take the order of the plies into account, the found designs for bending dominated problems are more conservative.

Keeping in mind that general design practice often requires laminates to consist of several fiber orientations, as well as accounting for multiple load cases, it can be concluded that for the purpose of practical design problems the predicted factor of safety will only be slightly conservative with respect to the Tsai–Wu failure criterion. Hence, the proposed design envelope is well suited to in-plane strength-optimization problems.

It can also be remarked that sandwich structures with composite face sheets are often used when bending loads are dominant. In such a case, the face sheets are primarily loaded in their plane. Thus, the design of the face sheets would reduce to an in-plane design problem, for which the proposed envelope is well suited.

B. Verification of the Strength Formulation

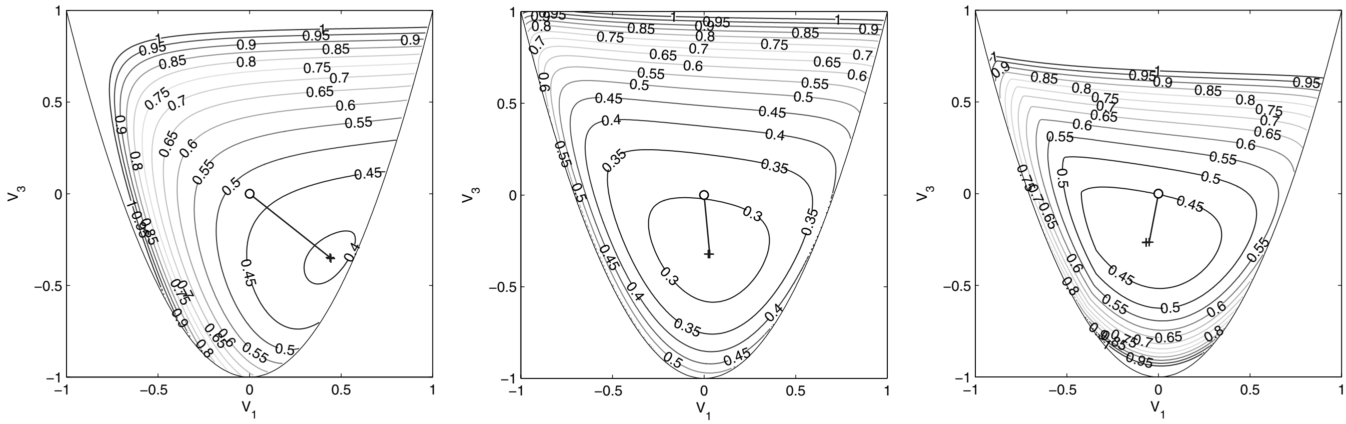
To verify the proposed formulation, the laminate strength is maximized for the three different materials for a combination of shear with either tensile or compressive loads, with $\omega = 0.50$. The results for an example set are plotted in Fig. 3 along with the contours of the function, r, which is being minimized, and the optimization path. The optimal values of the lamination parameters for each case are presented in Table 2. In each case, the optimization is started from $V_1 = V_3 = 0$, which corresponds to a quasi-isotropic laminate.

The nonsmooth strength contours in Fig. 3c are caused by the nonsmooth nature of the fourth-order envelope [Eq. (13)]. It is this lack of smoothness that suggested the use of a bound formulation for this case in Sec. V.A. The optimization procedure converged in a small number of iterations, typically less than 10 for a convergence tolerance of $1 \cdot 10^{-6}$ for the value of r. Often, good convergence to near optimal solutions can be achieved in one iteration, as can be observed from Fig. 3.

In Table 3, the optimal lamination parameters for a range of ω values are shown for the case of tension/shear loading. It is interesting to note that for carbon/epoxy (IM6) the optimal pure-tension design ($\omega = 0$) is an angle-ply laminate with fiber orientations of ± 5 deg. Brandmaier [28] also found that, as a function of the material properties, maximum strength under

Table 1 Ratio of the safety factor predicted by the design envelope and that obtained from the Tsai–Wu failure criterion ($\lambda_{\mathrm{TW}}/\lambda$)

	Axial tension (N_x)				Bending (M_x)	
Material	$[0_8]_s$	$[90_8]_s$	$[\pm 45_4]_s$	$[\pm 45, 0_4, 90_2]_s$	$[\pm 45, 0_4, 90_2]_s$	$[0_4, \pm 45, 90_2]_s$
Carbon/PEEK (AS4)	2.99	1.00	1.12	1.00	1.07	1.17
Carbon/epoxy (IM6)	3.92	1.19	1.33	1.12	1.20	1.05
Boron/epoxy (B5.6)	2.62	1.00	2.26	1.00	1.33	1.17



a) Carbon/PEEK (AS4) - Tension **b) Carbon/Epoxy (IM6) - Compression** **c) Boron/Epoxy (B5.6) - Compression**

Fig. 3 Sample optimization paths for the three investigated materials, with $\mathbf{N} = \pm N \cdot [0.5 0.0.5]^T$.

unidirectional loading may not be achieved by aligning fibers in the primary stress direction.

The developed failure envelope, and subsequent optimization formulation is applicable to design problems under in-plane and bending loads and combinations thereof. Pure bending loading does not differ in any essential way from the in-plane case. Combined in-plane and bending loading necessitates the use of both in-plane and out-of-plane lamination parameters. The feasible region in this case is available as an approximation [13] and, as such, the basic approach is still applicable albeit with a few changes in the details. Therefore, all of the following optimization results are limited to in-plane loading, which, the authors feel, is representative enough of the practical applications of the approach proposed in this paper.

C. Parametric Study

Because a general formulation has been developed for both strength- and stiffness-driven optima, it is interesting to investigate the difference between the two for a range of load cases. The results of the inverse of the safety factor, $1/\lambda = \sqrt{r}$, are plotted in Fig. 4 for the three material systems as a function of the axial load to shear load ratio for the different laminate constructions. The designs in Figs. 4a and 4b are symmetric balanced composites and, thus, require only two lamination parameters. The remaining two figures are for unbalanced laminates and, hence, all four in-plane lamination parameters are used during the optimization.

One of the first noticeable trends is the closeness of the strength- and stiffness-optimized designs for pure tension/compression or shear loads (i.e., $\omega = 0$ or $\omega = 1$). In fact, for balanced laminates they are extremely close, if not identical. Only in the case of unbalanced laminates, particularly for boron/epoxy (B5.6), is a significant difference between strength- and stiffness-optimized results observable.

As one might expect, the strength-optimized designs will always have a factor of safety (lower $\sqrt{r}$) higher than or equal to the stiffness-driven designs. The performance of combined tension and shear loaded (Figs. 4a and 4c) laminates is similar for both strength- and stiffness-based designs over the entire range of loading, with boron/epoxy (B5.6) deviating from this trend the most. For the combined compression and shear loaded cases (Figs. 4b and 4d), the difference between the stiffness- and strength-driven designs can be as much as 48%. Comparing balanced (Figs. 4a and 4b) with unbalanced designs (Figs. 4c and 4d) for the same load cases, it can be seen that the unbalanced designs yield a higher factor of safety for $\omega > 0$, that is, when shear loading is present.

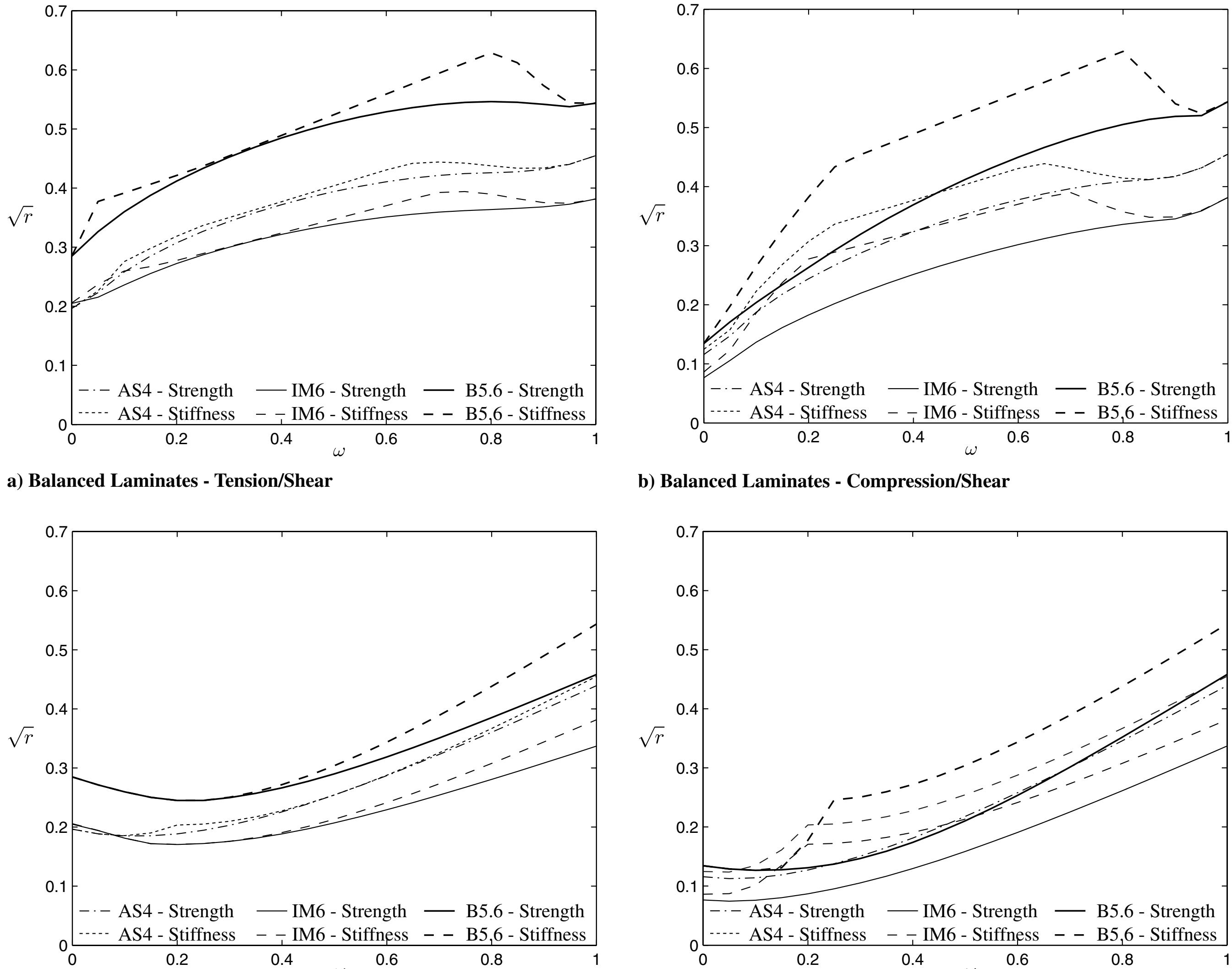
Because the formulation for maximum stiffness is insensitive to the sign of the loading, one would expect the stiffness optimization to produce the same design for both tension and compression. This cannot be discerned from Fig. 4, because the same set of lamination parameters will yield a different safety factor for tension and compression. However, the stiffness-optimized designs are found

Table 2 Optimum lamination parameters and $\sqrt{r}$ values for $\omega = 0.5$ for various materials and loading

	Tension/shear			Compression/shear		
Material	V_1	V_3	$\sqrt{r}$	V_1	V_3	$\sqrt{r}$
Carbon/PEEK (AS4)	0.444	−0.353	0.394	0.155	−0.357	0.353
Carbon/epoxy (IM6)	0.558	−0.293	0.339	0.023	−0.322	0.279
Boron/epoxy (B5.6)	0.601	−0.248	0.510	−0.067	−0.266	0.412

Table 3 Optimum lamination parameters and $\sqrt{r}$ values for a range of tension/shear load cases ($\omega = 0.1, \ldots, 1.0$)

	Carbon/PEEK (AS4)			Carbon/epoxy (IM6)			Boron/epoxy (B5.6)		
ω	V_1	V_3	$\sqrt{r}$	V_1	V_3	$\sqrt{r}$	V_1	V_3	$\sqrt{r}$
0.0	1.000	1.000	0.196	0.985	0.940	0.205	1.000	1.000	0.2847
0.2	0.692	0.305	0.307	0.796	0.324	0.272	0.794	0.261	0.4117
0.4	0.512	−0.163	0.372	0.626	−0.114	0.322	0.670	−0.092	0.4846
0.6	0.381	−0.531	0.411	0.494	−0.463	0.351	0.535	−0.397	0.5291
0.8	0.241	−0.884	0.426	0.307	−0.811	0.364	0.383	−0.707	0.5462
1.0	0.000	−1.000	0.455	0.000	−1.000	0.382	0.000	−1.000	0.5435



a) Balanced Laminates - Tension/Shear

b) Balanced Laminates - Compression/Shear

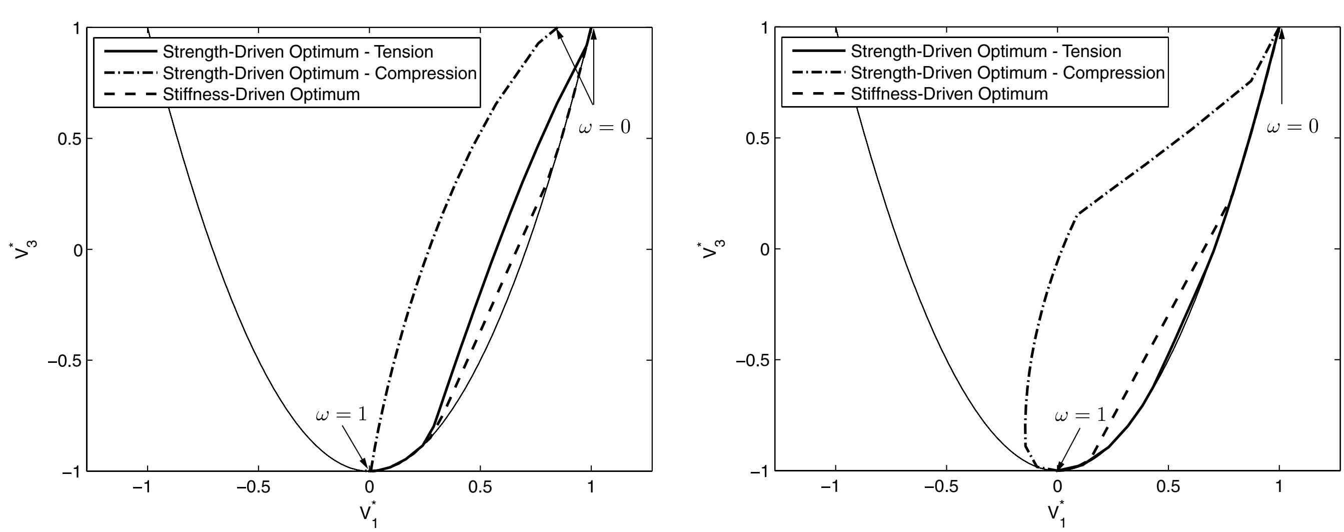
c) Unbalanced Laminates - Tension/Shear

d) Unbalanced Laminates - Compression/Shear

Fig. 4 Comparison of stiffness- and strength-driven optima for various materials.

to be identical and yield the same lamination parameters for both load cases.

When considering the balanced designs (Figs. 4a and 4b), it is also interesting to investigate the path of optimum design variables in the lamination parameter design space for the load cases considered, as is done for carbon/PEEK (AS4) and boron/epoxy (B5.6) in Fig. 5. The designs in the top right-hand corner of the lamination parameter design space correspond to the pure-tension or compression load



a) Carbon/PEEK (AS4)

b) Boron/Epoxy (B5.6)

Fig. 5 Optimization paths for strength- and stiffness-driven optima for balanced symmetric laminates.

cases ($\omega = 0$), whereas the lines at the bottom of the parabola correspond to the pure shear case ($\omega = 1$). It should be kept in mind that $V_1^* = 1$, $V_3^* = 1$ corresponds to a laminate with only 0 deg fibers, whereas $V_1^* = 0$, $V_3^* = -1$ corresponds to one with a ± 45 deg lay up. From these figures it is also clear that the designs for maximum compressive strength deviate more from the stiffness-driven optima than those for maximum tensile strength.

Looking at AS4 (Fig. 5a), pure tension and pure shear result in identical laminate lay ups, 0 and ± 45 deg, respectively, for both stiffness- and strength-driven optimization. However, in the case of pure compression, the strength-based design includes a small percentage (<5%) of 90 deg fibers. This improves the tensile strength in the direction perpendicular to the loading, reducing the laminate's Poisson ratio and, hence, increasing the laminate's strength slightly.

The results for boron/epoxy (B5.6) shown in Fig. 5b also indicate that ± 45 deg yields the best design in the case of pure shear loading. It can also be seen that for the tension/shear designs optimal lay ups are all angle-ply laminates. The large difference between the stiffness- and strength-driven optima for the compression/shear load case is also clearly visible.

The exact behavior of the optimal fiber orientation for maximum strength is a complex function of material properties and failure allowables [28]. The clear dependency on the individual material properties is to be expected due to the strong material dependence of the Tsai–Wu failure criterion [19]. Identifying which material properties or ratios thereof and their influence on the degree of correlation between stiffness- and strength-optimized designs will be the subject of future research.

VII. Conclusions

The implementation of the Tsai–Wu failure criterion in the lamination parameter design space is presented. Analytical expressions are found representing the conservative failure envelope in the strain space. The equations for the envelope are functions of only two strain invariants and do not depend explicitly on the stacking sequence. The active envelope equation can be used to formulate a failure index related to the factor of safety. The failure index is used to formulate the optimization problem of designing panels for maximum strength for both cases of pure in-plane loading and combined in-plane and bending loads. The derived envelope is shown to accurately represent the factor of safety of practical laminates under in-plane loading; however, for bending dominated problems it may be too conservative.

Panels under combined axial and shear loads are designed for maximum strength and stiffness. The strength-optimization problem is solved using a successive approximation methodology. An approximation of the failure index is proposed and implemented. Comparing the results of strength- and stiffness-driven optima for various materials and load conditions, it becomes apparent that the difference between the two optima depends strongly on the material properties and loading. Strength-driven optimization can lead to an increase of up to 48% in the factor of safety compared with stiffness-based designs. It is thus concluded that, although stiffness maximization might reasonably serve as an easier to evaluate surrogate for the preliminary design of composite structures, the derived conservative failure index offers a more attractive and easier to implement alternative.

Two open problems remain. The first is to investigate and understand how the shape of the conservative failure envelope depends on the material stiffness properties and failure stresses. Such an analysis would also be quite useful for extending the present formulation to more complex failure criteria that differentiate between different failure modes, such as Hashin's criterion. The second is the convexity of the failure index function in the lamination parameter space. Convexity would mean that strength-optimal designs are unique, and that this unique optimum can be converged starting from arbitrary initial designs. Numerical experience seems to indicate that the failure index is indeed convex or nearly so. Although the convexity of the failure index in strain space is straightforward to demonstrate, whether or not the convexity carries over to the lamination parameter space is more challenging to assess.

Appendix A: Material Invariants

The material invariants, U_i for $i = 1, \ldots, 5$, are defined in terms of lamina reduced stiffnesses by [22]:

$$U_1 = (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})/8$$

$$U_2 = (Q_{11} - Q_{22})/2, \qquad U_3 = (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})/8$$

$$U_4 = (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})/8$$

$$U_5 = (Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})/8$$

The coefficients for the Tsai–Wu failure criterion [Eq. (5)] are given by

$$F_{11} = \frac{1}{X_t X_c} \qquad F_{22} = \frac{1}{Y_t Y_c} \qquad F_1 = \frac{1}{X_t} - \frac{1}{X_c}$$

$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \qquad F_{12} = \frac{-1}{2\sqrt{X_t X_c Y_t Y_c}} \qquad F_{66} = \frac{1}{S^2}$$

where X_c, X_t, Y_c, Y_t, and S are the failure stresses in compression, tension, and shear in the principle material directions. F_{12} is known as the interaction coefficient [16] and can be approximated with the equivalent von Mises yield criterion for composites. The material stresses and strains can be related via the stiffness matrix, yielding the coefficients for the strain equivalent of the Tsai–Wu failure criterion [Eq. (6)], given by:

$$G_{11} = Q_{11}^2 F_{11} + Q_{12}^2 F_{22} + 2F_{12} Q_{11} Q_{12}$$

$$G_{22} = Q_{12}^2 F_{11} + Q_{22}^2 F_{22} + 2F_{12} Q_{12} Q_{22}$$

$$G_1 = Q_{11} F_1 + Q_{12} F_2 \qquad G_2 = Q_{12} F_1 + Q_{22} F_2$$

$$G_{12} = Q_{11} Q_{12} F_{11} + Q_{12} Q_{22} F_{22} + F_{12} Q_{12}^2 + F_{12} Q_{11} Q_{22}$$

$$G_{66} = 4Q_{66}^2 F_{66}$$

Additionally, when solving for the feasible design region, the formulation can be simplified by defining the following materials invariants:

$$u_1 = G_{11} + G_{22} - 2G_{12}, \qquad u_2 = (G_1 + G_2)/2$$

$$u_3 = (G_{11} + G_{22} + 2G_{12})/4, \qquad u_4 = G_1 - G_2$$

$$u_5 = G_{11} - G_{22}, \qquad u_6 = G_{66}$$

Appendix B: Material Properties

Table B1 Tabulation of the properties of the materials mentioned in this paper [16]

	Carbon/ PEEK (AS4)	Carbon/ epoxy (IM6)	Boron/ epoxy (B5.6)
Longitudinal modulus (E_1, GPa)	142.0	177.0	201.0
Transverse modulus (E_2, GPa)	10.3	10.8	21.7
Shear modulus (G_{12}, GPa)	7.2	7.6	5.4
Poisson's ratio (ν_{12})	0.27	0.27	0.17
Longitudinal tensile strength (X_t, MPa)	2280.0	2860.0	1380.0
Longitudinal compressive strength (X_c, MPa)	1440.0	1875.0	1600.0
Transverse tensile strength (Y_t, MPa)	57.0	49.0	56.6
Transverse compressive strength (Y_c, MPa)	228.0	246.0	125.0
Shear strength (S, MPa)	71.0	83.0	62.6

Appendix C: Strength Optimization Coefficients

To determine λ, two polynomial equations, $f_1(I_1, I_2)$ and $f_1(I_1, I_2)$, were defined [Eq. (16)], for which the coefficients are given by

$$a_{10} = u_4^2 + 4u_1 - 4u_6, \qquad a_{11} = -4u_2 I_1(u_1 - u_6) + 2u_4 u_5 I_1$$

$$a_{12} = 4u_6^2 I_2^2 - 4u_3 I_1^2(u_1 - u_6) - 4u_6 u_1 I_2^2 + u_5^2 I_1^2, \qquad a_{20} = 1$$

$$a_{21} = -2u_2 I_1, \qquad a_{22} = -2u_3 I_1^2 + u_2^2 I_1^2 - I_2^2\left(u_4^2 + 2u_1\right)$$

$$a_{23} = 2u_2 I_1^3 u_3 - I_2^2(2u_4 u_5 I_1 - 2u_1 u_2 I_1)$$

$$a_{24} = u_1^2 I_2^4 - I_2^2\left(u_5^2 I_1^2 - 2u_1 u_3 I_1^2\right) + u_3^2 I_1^4$$

where I_1 and I_2 are the strain invariants, and the coefficients u_i are the material invariants defined in Appendix A. Additionally, when optimizing for strength, the Hessian matrix $\mathbf{H}$ in Eq. (26) can be derived in terms of the strain invariants using the chain rule

$$\frac{\partial^2 r}{\partial\epsilon_k \partial\epsilon_l} = \sum_{p=1}^{2}\left[\sum_{q=1}^{2}\left[\left(\frac{\partial^2 r}{\partial\lambda^2}\cdot\frac{\partial\lambda}{\partial I_p}\cdot\frac{\partial\lambda}{\partial I_q} + \frac{\partial r}{\partial\lambda}\cdot\frac{\partial^2\lambda}{\partial I_p \partial I_q}\right)\frac{\partial I_q}{\partial\epsilon_k}\right]\frac{\partial I_p}{\partial\epsilon_l} + \frac{\partial r}{\partial\lambda}\cdot\frac{\partial\lambda}{\partial I_p}\cdot\frac{\partial^2 I_p}{\partial\epsilon_k\partial\epsilon_l}\right]$$

where $k, l = 1, \ldots, 3$. The derivatives of the strain invariants with respect to the strains can be readily found by differentiation of Eq. (14). The derivative of λ with respect to the strain invariants is found by making use of the equations derived for the failure envelope [Eqs. (16)], which can be rewritten as

$$f_i(\lambda) = \sum_{n=0}^{N} a_n \lambda^n = 0 \qquad \text{(C1)}$$

where $N = 2$ or 4 and is related to the applicable failure envelope for $i = 1$ or 2 for which the corresponding coefficients a_n are already defined. To determine the derivative of λ, the equations listed can be differentiated with respect to I_p, which results in

$$\sum_{n=0}^{N}\left[\frac{\partial a_n}{\partial I_p}\lambda^n + a_n\frac{\partial\lambda^n}{\partial I_p}\right] = 0$$

Therefore,

$$\frac{\partial\lambda}{\partial I_p} = -\frac{\sum_{n=1}^{N}\frac{\partial a_n}{\partial I_p}\lambda^n}{\sum_{n=1}^{N} n a_n \lambda^{n-1}} \qquad \text{(C2)}$$

The second-order derivatives of λ can be found in a similar fashion by taking the derivative of Eq. (C2) with respect to the strain invariant I_q.

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